A Recursive View of Skydiving

Adapted from *Navigating Through Discrete Mathematics in Grades 6-12*, NCTM, 2008.

Skydiving is an exciting but dangerous sport. Many precautions are taken to ensure the safety of the skydivers. The basic fact underlying these precautions is that acceleration due to the force of gravity is 32 feet per second per second (written as 32 ft/sec^2). Thus, each second that the skydiver is falling, her speed increases by 32 ft/sec. (Throughout this investigation we ignore air resistance and other complicating factors; we focus only on the force of gravity.)

As you work on this task, look for answers to these questions:

- *What are recursive and explicit formulas for the total distance fallen by a skydiver after each second before her parachute opens?*

- *What methods can you use to find these formulas?*

- *How are quadratic functions involved and why?*

To help answer these questions, consider the following table, which you will complete in the problems below.

A Skydiver's Speed and Distance Fallen Before the Parachute Opens

Time in seconds n	Instantaneous Speed at time n	Average Speed during each second	Distance Fallen during each second $D(n)$	Total Distance Fallen after n seconds $T(n)$
0	0	0	0	0
1 sec	32 ft/sec	16 ft/sec	16 ft	16 ft
2 sec	64 ft/sec	48 ft/sec	48 ft	64 ft
3 sec				
4 sec				
:	:	:	:	:
n sec				

1. Explain each entry in the row corresponding to Time = 1 sec in the table above. (The basis for computing all entries is the fact that acceleration due to gravity is 32 ft/sec^2.)

2. Explain each entry in the row corresponding to Time = 2 sec.

3. Complete the table for Time = 3 sec and Time = 4 sec. Compare your table entries to those of some of your classmates. Discuss and resolve any differences.

Distance Fallen During Each Second – Now you will use the completed table to help find formulas for distance fallen. First, consider distance fallen during each second.

4. Find recursive and explicit formulas for D(n), Distance Fallen during the nth second, as follows. (For this problem, ignore the row in the table for Time = 0 sec.)

    a. If NOW is the Distance Fallen during any given second and NEXT is the Distance Fallen during the next second, write an equation for NEXT in terms of NOW.

    b. Rewrite the NEXT/NOW equation using D(n) and D(n-1).
    That is, if D(n) = Distance Fallen during the nth second, and D(n-1) = Distance Fallen during the (n-1)st second, write an equation for D(n) in terms of D(n-1). (This is a ***recursive formula*** since D(n) is expressed in terms of a previous value, D(n-1).)

    c. If D(n) = Distance Fallen during the nth second, write an equation for D(n) in terms of n. Explain how you got your equation and why it is correct.
    (A formula like this, where D(n) is written as a function of n, is called an ***explicit* or *closed-form formula*.)

Total Distance Fallen After *n* Seconds – The main goal of this investigation is to find formulas for T(n), the Total Distance Fallen after *n* seconds. You will use several methods to do this:

*   general analysis
*   use an arithmetic sequence
*   use a finite differences table and solve a linear system with matrices
*   use a finite differences table and solve a linear system without matrices

Each of these methods is carried out in the next four problems.

5. Find formulas for Total Distance Fallen after *n* seconds, as follows. Let T(*n*) = Total Distance Fallen after the nth second.

    a. As part of Problem 3, you computed T(3). Describe how you computed T(3).

    b. Describe all the methods you can think of for how to compute T(*n*).

    c. Write a formula for T(*n*) in terms of T(*n*-1) and D(*n*). (This is a *recursive formula* since T(n) is expressed in terms of the previous value, T(*n*-1).)

6. Find an explicit formula for T(*n*) by summing an arithmetic sequence, as follows. (This problem assumes previous familiarity with arithmetic sequences.)

    a. One way that you may have described in Problem 5b for finding T(*n*) is to sum all the terms up to D(*n*) in the D(*n*) column. If you didn't already describe this in Problem 5b, explain here why this is a valid method for computing T(*n*).

    b. You found in Problem 4c above that D(n) = D(n-1) + 32 ft (ignoring the row for Time = 0 sec). This formula shows that you add a constant, 32, each time to get the next value of D(n). Thus, the terms D(n) form an arithmetic sequence. Therefore, T(n) = the sum of the arithmetic sequence: D(1) + D(2) + … + D(n). Compute this sum to find an explicit formula for T(n) in terms of n.

7. Another way to find an explicit formula for T(n) is to use what is called a *finite differences table*. Here's how it works.

    a. Complete the three remaining entries in the bottom of the table below.

Finite Differences Table

n	T(n)	1st Differences (entry in the previous column) – (entry just above it)	2nd Differences (entry in the previous column) – (entry just above it)
1	16	---------------	---------------
2	64	64 - 16 = 48	---------------
3	144	144 – 64 = 80	80 – 48 = 32
4	256	256 – 144 = 112	
5	400		

    b. Describe the pattern in the 2nd differences column.

    c. Now we apply a key fact: If the nth differences in a finite differences table are constant, then the formula for T(n) is an nth-degree polynomial. In this case, the 2nd differences are constant, so the formula for T(n) is a 2nd-degree polynomial, that is, the formula is quadratic. (Proving this key fact is not too hard, but it will take too long to do it now. Your teacher might give you guidance or references for the proof if you are interested.)

So we know that T(n) is quadratic and thus it looks like:

$$T(n) = an^2 + bn + c$$

Now we need to find the coefficients, a, b, and c. One way to find a, b, and c is to generate and solve a system of three linear equations. To help us do this, we know the value of T(n) for several values of n. Thus we get:

$T(n) = an^2 + bn + c$

n=1 → $16 = a + b + c$

n=2 → $64 = 4a + 2b + c$

n=3 → $144 = 9a + 3b + c$

Explain the details of how these three equations are generated.

    d. Now you need to solve this system of three linear equations. One way to do so is by using matrices. (This problem involves previous familiarity with matrices. For another method that does not use matrices, see Problem 8 below.) To begin, a system of linear equations like this can be represented using matrices, as follows:

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = \begin{vmatrix} 16 \\ 64 \\ 144 \end{vmatrix}$$

Explain where all the entries in the matrices come from, and why this matrix equation is equivalent to the linear system in Part c.

e. You can solve this matrix equation by multiplying both sides of the equation on the left by the inverse of matrix $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}$.

This gives you: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 64 \\ 144 \end{bmatrix}$.

Find this inverse matrix and carry out the multiplication to solve the matrix equation. (You may want to use your calculator to carry out these computations.) What are the values for a, b, and c?

f. Using the values for a, b, and c that you just found, what is the formula for T(n)?

g. Check the formula you found in Part f by evaluating it for some values of n, and verifying that you get the same values for T(n) as in the tables above.

8. Another way to solve systems of linear equations like the system in Problem 7 is to use algebra without matrices. To do this, you need to combine and manipulate the three equations in Part c until you can solve for a, b, and c. The combining and manipulating is similar to what you do for a system of two linear equations, but more complicated since there are more equations. Try this method. Check that you get the same solution as in Part f. (For an example of how this is done in a similar problem, see *Mission Mathematics, Grades 9-12*, NCTM, p. 23.)

Summary

a. What are recursive and explicit formulas for T(n), the total distance fallen by the skydiver?

b. Describe the methods you used for finding the formulas in Part a.

c. The explicit formulas that you found for T(n) are quadratic functions. Compare patterns of the quadratic functions that you have worked with in this investigation to patterns of linear and exponential functions you have studied previously, as follows:

- Examine the list of values for T(n) shown in the table at the beginning of this investigation (the last column in the table on page 1). How is the pattern of change shown in the list of values for T(n) different from the pattern of change in tables for linear and exponential functions?

- How is the recursive formula you found for T(n) in Problem 5c different from recursive formulas for linear and exponential functions?

- How are the explicit formulas you found for T(n) in Problems 6, 7, and 8 different from the explicit formulas for linear and exponential functions?

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